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Discrete Prolate Spheroidal Sequence

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Abbreviations

DFT	Discrete Fourier Transform
DPSS	Discrete Prolate Spheroidal Sequence
MTM	Multitaper Method

Definition

The *discrete prolate spheroidal sequences (DPSSs)* are maximally concentrated in both the time and frequency domains. This is a crucial property for applications in power spectrum estimation. The DPSSs comprise sets of real-valued orthonormal sequences (discrete functions) with the following properties: (i) They are limited within the spectral band $[-W, W]$, where $W > 0$. (ii) The total energy of the sequence (i.e., the sum of the squares of the amplitudes) over a finite time interval is maximized. The *time-limited DPSSs,* also known as the *Slepian sequences*, are orthonormal sequences with the following properties: (i) They are *time-limited* within a finite time interval. (ii) They exhibit maximal spectral concentration in the frequency band $[-W, W]$. The DPSSs are parametrized in terms of the length N of the temporal sequence, the bandwidth W, and the order $k = 0, 1, \ldots N - 1$ of the Slepian sequence.

Overview

In Fourier analysis, it is known that functions cannot be fully localized in both time and frequency. As an extreme example, consider the *cosine* function $\cos(2\pi f_0 t)$ with frequency f_0. Its Fourier transform involves the sum of two Dirac delta functions, $\delta(f \pm f_0)$. The *cosine* is fully extended in time, but the delta functions are completely localized at $\pm f_0$. In less extreme cases, functions that are concentrated (localized) in either the time or frequency space are extended (delocalized) in the other. This trade-off between localization in the direct (space or time) versus the spectral domain is also evident in Heisenberg's uncertainty principle of quantum physics.

Since functions cannot be completely confined both in the temporal and spectral domains, a fundamental problem is how to optimally concentrate the energy of a signal in the time (frequency) domain if the signal is spectrally (temporally) confined. The optimal concentration problem was addressed in a series of papers by the mathematicians Slepian, Pollack, and Landau (Landau and Pollak 1961, 1962; Slepian and Pollak 1961; Slepian 1978, 1983). These theoretical advances established the framework for the development of powerful spectral estimation methods (Thomson 1982; Percival and Walden 1993). The DPSSs solve the maximal concentration problem in the case of discrete functions (i.e., for time series). The solution of the concentration problem for continuous functions is provided by the so-called *prolate spheroidal wave functions* (Slepian 1978). The DPSS development is reviewed by Slepian (1983).

Methodology

Consider a finite-length discrete time sequence $t_n = n\delta t$, $n = 0, 1, \ldots, N - 1$, where $N > 1$ and the time step is $\delta t > 0$. In addition, let $x(t)$ represent a square-summable (finite-energy) signal, sampled at the times t_n, i.e., $x_n = x(t_n)$, $n = 0, \ldots, N - 1$. The *discrete-time Fourier transform (DFT)* of the sequence is given by

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$$\widetilde{x}(f) = \sum_{n=0}^{N-1} \exp\left(-2\pi i f t_n\right) x_n. \quad (1)$$

The DFT is defined over the frequency interval $-f_N < f \leq f_N$, where $f_N = f_s/2$ is the *Nyquist frequency* and $f_s = 1/\delta t$ is the sampling frequency.

Formulation of the Concentration Problem

The concentration problem consists of finding all the finite-energy time sequences $\{x_n\}_{n=0}^{N-1}$ which maximize the *spectral concentration ratio*

$$\lambda = \frac{\int_{-W}^{W} |\widetilde{x}(f)|^2 df}{\int_{-f_N}^{f_N} |\widetilde{x}(f)|^2 df}, \quad \text{where } 0 < W < f_N. \quad (2)$$

The parameter λ, $0 \leq \lambda \leq 1$, measures the ratio of the energy contained within the spectral band $[-W, W]$ over the total energy. The DPSSs comprise N discrete functions, denoted by $\upsilon_n^{(k)}(W,N)$, $k = 0, 1, \ldots, N-1$, that maximize λ and are also orthogonal to each other. The DPSSs are the real-valued solutions of the following set of equations (Slepian 1978)

$$\sum_{m=0}^{N-1} \frac{\sin 2\pi W(n-m)}{\pi(n-m)} \upsilon_n^{(k)}(W,N) = \lambda_k(N,W)\, \upsilon_n^{(k)}(W,N), \quad n = 0,1,\ldots,N-1. \quad (3)$$

The above system is equivalent to the following eigenvalue problem:

$$\mathbf{A}\mathbf{v}^{(k)} = \lambda_k \mathrm{v}^{(k)}. \quad (4)$$

In Eq. (4), $\mathbf{A}$ is the *prolate matrix* with elements

$$A_{n,m} = \frac{\sin 2\pi W(n-m)}{\pi(n-m)}, \quad n, m = 0, \ldots, N-1,$$

$\{\lambda_k\}_{k=0}^{N-1}$ is the set of N eigenvalues, and the DPSS vectors $\mathrm{v}^{(k)} = \left(\upsilon_0^{(k)}, \ldots, \upsilon_{N-1}^{(k)}\right)^{\top}$ are the respective eigenvectors ("⊤" denotes the transpose), which form the *Slepian basis*. The eigenvalues λ_k represent the spectral concentration ratios for the respective eigenvectors $\mathrm{v}^{(k)}$.

For continuous functions of time, the concentration problem is solved by an integral equation whose eigenfunctions are the *discrete prolate spheroidal wavefunctions* (Slepian and Pollak 1961; Slepian 1978).

Properties of the Slepian Eigenvectors and Eigenvalues

The DPSSs possess certain useful mathematical properties (Slepian 1983; Percival and Walden 1993; Lii and Rosenblatt 2008):

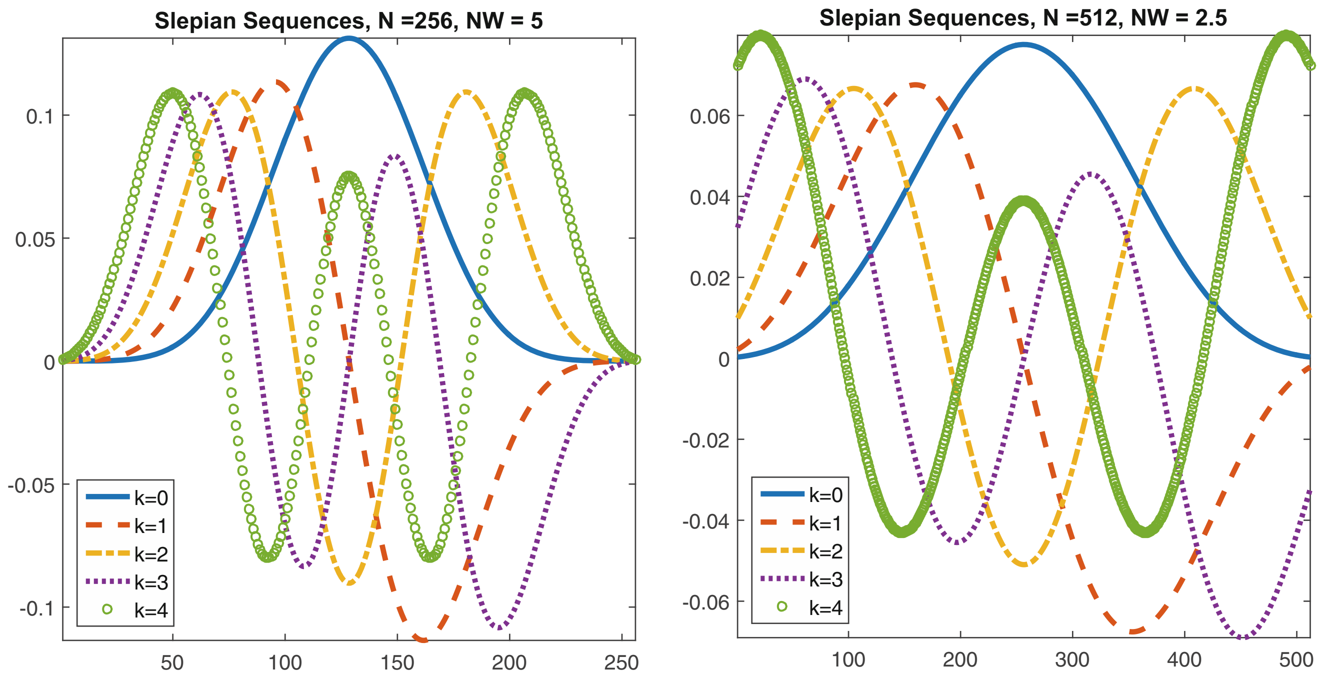
1. The eigenvalues are distinct, real-valued, and positive: $1 > \lambda_0 > \lambda_1 > \ldots \lambda_{N-1} > 0$. The positive eigenvalues reflect the positive-definiteness of the prolate matrix $\mathbf{A}$.
2. The eigenvectors $\mathrm{v}^{(k)}$ are even or odd according to whether k is even or odd.
3. The eigenvector $\mathrm{v}^{(k)}$ has exactly k zeros in the open interval $(0, N-1)$.
4. The eigenvectors that correspond to different eigenvalues are mutually orthogonal, i.e., each eigenvector $\mathrm{v}^{(k)}$ is orthogonal to all eigenvectors $\mathrm{v}^{(l)}$, $l = 0, \ldots, k-1$.
5. The eigenvalues λ_k exhibit *clustering behavior*: Slightly fewer than $N(W/f_N)$ eigenvalues are very close to one, slightly fewer than $N(1 - W/f_N)$ eigenvalues are very close to zero, while the remaining (very few) eigenvalues are not close to either one or zero (Lii and Rosenblatt 2008). Hence, the number $N(W/f_N)$ (which equals $2NW$, if $\delta t = 1$) is a rough estimate (upper bound) of the number of significant eigenvalues.
6. The integer part of $2NW$ is an approximate estimate for the dimension of the space of functions that are "essentially concentrated" in both time and frequency.

The first five eigenvectors for two Slepian bases displayed in Fig. 1 illustrate the above properties. The plot on the left-hand side of Fig. 1 corresponds to $N = 256$, $NW = 5$, and $\delta t = 1$. The five respective eigenvalues λ_k are all very close to one. The plot on the right-hand side corresponds to $N = 512$, $NW = 2.5$, and $\delta t = 1$. The respective eigenvalues λ_k are given by 1, 0.99984, 0.99622, 0.95213, and 0.71389. Note that λ_0 appears as 1 because $1 - \lambda_0 \approx 3 \cdot 10^{-6}$.

Applications in Spectral Estimation

An important practical problem is the estimation of the *power spectrum (power spectral density)* of a process based on a finite set of observations (sample) of the process.

The following focuses on *wide-sense stationary, ergodic random processes* $Z(t)$, which are assumed to extend over the infinite time domain, i.e., $t \in (-\infty, \infty)$. The spectral density $S(f)$ of $Z(t)$ is well-defined and coincides with the Fourier transform of the autocorrelation function according to the Bochner-Khinchin-Wiener theorem. The time series $\{z_n\}_{n=0}^{N-1}$, where $z_n = z(t_n)$, represents a finite sample of $Z(t)$ ("z" is used herein to avoid confusion with maximally concentrated functions denoted by "x").



Discrete Prolate Spheroidal Sequence, Fig. 1 The first five DPSS eigenvectors for $N = 256$, $W = 5/N$ (left plot) and $N = 512$, $W = 2.5/N$ (right plot). Horizontal axes: time sequence ($\delta t = 1$). Vertical axes: DPSS sequences $\upsilon_n^{(k)}(W, N)$

Periodogram Estimator

The two-sided *periodogram estimate* $\widehat{S}(f)$ of the spectral density based on $\{z_n\}_{n=0}^{N-1}$ is given by

$$\widehat{S}(f) = \frac{\delta t}{N}\left|\sum_{n=0}^{N-1} z_n \exp\left(-2\pi i f t_n\right)\right|^2, - f_N < f \leq f_N. \quad (5)$$

To obtain the one-sided estimate (for $0 \leq f \leq f_N$), $\widehat{S}(f)$ in Eq. (5) is multiplied by two for all frequencies except for $f = 0$ and $f = f_N$.

An estimator is called *consistent* if its variance tends to zero as $N \to \infty$. Unfortunately, the periodogram $\widehat{S}(f)$ is not consistent, because its standard deviation is as large as the expectation (mean), even as the sample size $N \to \infty$.

The periodogram also suffers from *spectral leakage*. This occurs because the finite observation window implies that $Z(t)$ is multiplied by a boxcar kernel (rectangular time window) which localizes the signal within the finite sampling interval. Multiplication with the window function implies the convolution of the Fourier transform $\tilde{z}(f)$ with that of the boxcar kernel in the spectral domain; the convolution perturbs $\tilde{z}(f)$ by mixing values at neighboring frequencies.

Welch's Method

The method proposed by Welch reduces the variance of $\widehat{S}(f)$ by dividing the sample into different segments which are allowed to overlap. A modified periodogram (see below) is estimated for each time segment, and the segment estimates are averaged to generate the Welch estimate of the power spectral density. The assumed stationarity of $Z(t)$ ensures that the modified periodograms represent approximately uncorrelated estimates of $S(f)$. Thus, averaging over different segments reduces the variance of the Welch estimate.

The *modified periodograms* involve the multiplication of the time series (signal) in each time segment with a *window function*. The latter vanishes outside the specific segment, peaks in the middle of this segment, and is symmetric around the peak. Windows thus suppress the values of the time series near the segments' edges. Overlapping segments prevent loss of information, since signal values near the edges of one window are also near the center of neighboring windows. In addition, windowing reduces potential inter-segment correlations that could arise due to segment overlap. Various windows with subtle differences that impact *spectral leakage* are used in the literature, including the Slepian zero-order DPSS. The Kaiser (also known as Kaiser–Bessel) window provides a simple approximation of the Slepian window based on Bessel functions.

Multitaper Method for Spectral Density Estimation

The *multitaper method (MTM)* was also developed to address spectral leakage in power spectrum estimation (Thomson 1982; Percival and Walden 1993). In MTM, the DPSSs are used to construct low-bias, statistically consistent spectral estimators that reduce spectral leakage.

The MTM is a *nonparametric* estimation method, because it is data driven and does not need a parametric model of the process. For example, the classical periodogram and Welch's methods are also nonparametric, while the *maximum entropy method* is parametric (Percival and Walden 1993).

The variance reduction of the spectral estimates is achieved in MTM by using a small set of mutually orthogonal windows (tapers), while Welch's modified periodogram uses a single data taper. The optimal MTM tapers consist of DPSS eigenvectors, which minimize spectral leakage by maximizing energy concentration in the main spectral lobe. Both the mutual orthogonality and the optimal time-frequency concentration are critical for the success of the multitaper technique.

The MTM estimate is obtained by averaging K modified periodogram estimates based on a respective set of K Slepian tapers. Averaging reduces the variance of the power spectrum. Only the fraction of tapers $\left(\sim 2N\frac{W}{f_N}\right)$ that generate small spectral leakage are used. Asymptotic properties for the bias and variance of MTM spectral density estimates are studied in Lii and Rosenblatt (2008).

Let $S_k(f)$ denote the *modified periodogram* obtained with the k-th Slepian sequence, $\mathrm{v}^{(k)}$:

$$S_k(f) = \delta t \left| \sum_{n=0}^{N-1} \upsilon_n^{(k)} z_n \exp\left(-2\pi i f t_n\right) \right|^2. \tag{6}$$

In the simplest MTM version, the power spectral density is estimated by averaging the K modified periodograms:

$$S_{\mathrm{MTM}}(f) = \frac{1}{K} \sum_{k=0}^{K-1} S_k(f). \tag{7}$$

Other MTM versions use taper-dependent weights w_k (Percival and Walden 1993):

$$S_{\mathrm{MTM}}(f) = \frac{\sum_{k=0}^{K-1} w_k S_k(f)}{\sum_{k=0}^{K-1} w_k}. \tag{8}$$

Both the MTM and Welch's method average over approximately uncorrelated estimates of the power spectral density. However, the two approaches differ with respect to how they decorrelate the modified periodograms. Welch's method uses different segments of the signal for each periodogram. MTM uses the entire signal, but the decorrelation is enforced by the orthogonality of the Slepian tapers.

Selection of the MTM Parameters

The MTM involves two competing parameters: the number of tapers K and the half-bandwidth W. In order to understand their role, recall that DFT has a frequency resolution equal to the *Rayleigh frequency* $f_R = f_s/N$; this is the minimum frequency difference that can be resolved by the DFT of a finite-duration signal.

For MTM estimation, a large value of K is desirable to reduce the estimate for the variance of the power spectrum. However, in practice, only the first $K = N(W/f_N) - 1$ of the DPSS eigenvectors provide negligible spectral leakage (Slepian 1978; Thomson 1982; Ghil et al. 2002). Hence, if p is the largest integer that does not exceed $N(W/2f_N)$, K is bounded by $K \leq 2p - 1$. On the other hand, the frequency resolution of the MTM is $f'_R \approx p f_R = W$. This implies a trade-off between spectral resolution and variance reduction. Optimal values for p and K thus depend on the length N of the time series and the desired spectral resolution.

DPSS in the Geosciences

The Discrete Prolate Spheroidal Sequences find numerous applications in the spectral estimation of climatic time series, geochemical, paleoclimatic, oceanographic, stratigraphic, and seismological data [Percival and Walden (1993), Ghil et al. (2002) and references therein]. Extension to spherical Slepian functions for spatial and spectral concentration problems on the sphere has applications in geodesy, geophysics, and oceanography (Simons and Wang 2011).

Software Implementations

In Matlab and R software, the DPSS is implemented by means of functions called dpss; multitaper power spectral density estimation is implemented by means of the pmtm function in Matlab and the package multitaper in R. In Python, this functionality can be found in the spectrum module.

Summary or Conclusions

Discrete prolate spherical (Slepian) sequences provide a mathematically elegant solution to the problem of time-frequency concentration: They are limited within a specific time (spectral) window and also maximize the energy content within a finite spectral (time) window. Slepian sequences are used as window functions in Welch's modified periodogram spectral estimation. The DPSSs are also an integral component of the multitaper method of spectral estimation. The MTM leads to considerable improvements over the classical periodogram in terms of variance and spectral leakage reduction.

Cross-References

- ▶ Discrete Fourier Transform
- ▶ Maximum Entropy
- ▶ Power Spectral Density
- ▶ Spectral Analysis
- ▶ Stochastic Process
- ▶ Time Series Analysis

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