Covariance Models Based on Local Interaction (Spartan) Functionals

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SSRF Covariance Models

Motivation
Local interaction models
SSRF Covariance Functions
KL Expansions
Inverse Covariance Kernel
Fast interpolation
Conclusions
References

Outline

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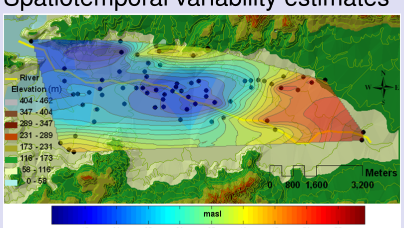
Environmental Applications

- Need for flexible positive-definite kernel functions *[Genton, 2002]*
- Faster interpolation and simulation methods for "big data"

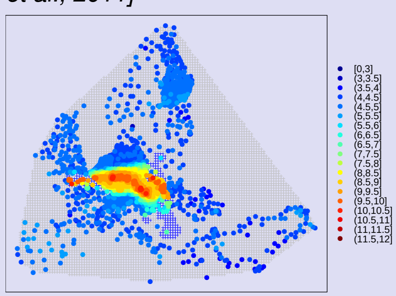
Groundwater monitoring

Groundwater level estimation in sparsely monitored basins *[Varouchakis and Hristopulos, 2013]*; Spatiotemporal variability estimates



Response to environmental threats

Radioactivity monitoring and emergency warning system *[Dubois et al., 2011]*



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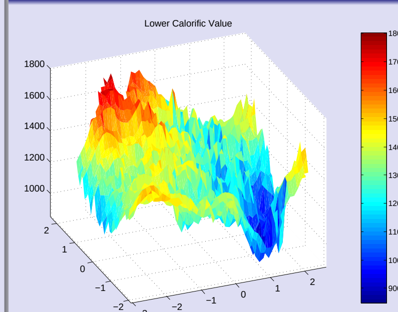
Applications in Mineral Reserves Estimation

Coal reserves & quality

Modeling of complex geological structures: Interpolation and simulation

Estimated lower calorific value (kcal/kg) — Amyndeo Mine (Western Macedonia)



Fluctuation-Gradient-Curvature (FGC) SSRF

- Gibbs probability density function (PDF)

$$f[\phi(\mathbf{s})] = \frac{e^{-\mathcal{H}[\phi(\mathbf{s})]}}{Z}, \quad \mathcal{H}[\phi(\mathbf{s})]: \text{ energy functional,}$$

- FGC energy functional - for simplicity assume $\mathbb{E}[\phi(\mathbf{s})] = 0$

$$\mathcal{H}_{\rm fgc}[\phi(\mathbf{s})] = \frac{1}{2\eta_0 \xi^d} \int_{\mathcal{D}} d\mathbf{s} \left\{ [\phi(\mathbf{s})]^2 + \eta_1 \xi^2 \, [\nabla \phi(\mathbf{s})]^2 + \xi^4 \left[\nabla^2 \phi(\mathbf{s})\right]^2 \right\}$$

- Correlation (covariance) function: $G(\mathbf{r}) = \mathbb{E}[\phi(\mathbf{s}+\mathbf{r})\, \phi(\mathbf{s})]$
- Properties: *Gaussian, zero-mean, stationary, isotropic SRF*

FGC-SSRF Coefficients

η_0 : scale, η_1 : stiffness, ξ: characteristic length; k_c : spectral cutoff

FGC-SSRF Covariance & Spectral Density

- Covariance: $G(\mathbf{r}) = \mathbb{E}[\phi(\mathbf{s})\, \phi(\mathbf{s}+\mathbf{r})]$. Fourier transform pair:

$$\tilde{G}(\mathbf{k}) = \int d\mathbf{r}\, e^{-\jmath \mathbf{k}\cdot\mathbf{r}}\, G(\mathbf{r}), \qquad G(\mathbf{r}) = \frac{1}{(2\,\pi)^d} \int d\mathbf{k}\, e^{\jmath \mathbf{k}\cdot\mathbf{r}}\, \tilde{G}(\mathbf{k}).$$

- Covariance spectral density:

$$\tilde{G}(k) = \frac{\mathbb{1}_{k_c \geq \kappa}(\kappa)\, \eta_0 \xi^d}{1 + \eta_1 \kappa^2 \xi^2 + \kappa^4 \xi^4}, \quad \kappa = \|\mathbf{k}\|, \quad \mathbb{1}_B(\cdot): \text{indicator function,}$$

- Permissibility conditions *(Bochner's theorem)* Hristopulos [2003]:

For any k_c:

$\eta_0 > 0, \xi > 0, \eta_1 > -2$

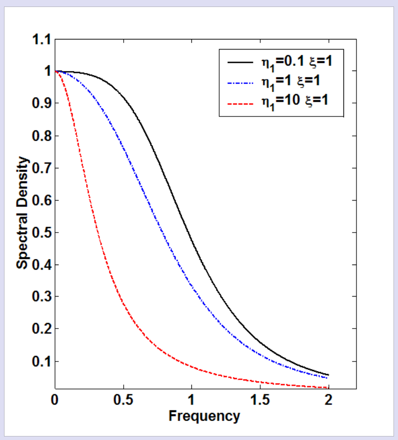
For finite k_c:

$\eta_1 < -2$, if $k_c \xi < \sqrt{\frac{|\eta_1| - \Delta}{2}}$

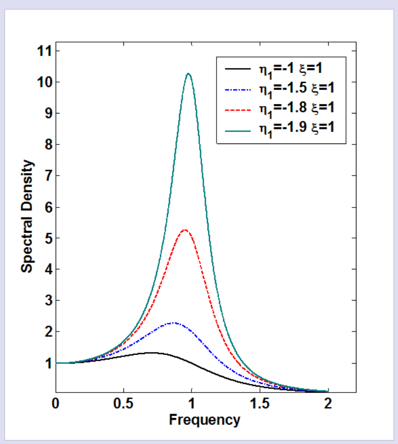
$\Delta = \sqrt{\eta_1^2 - 4}$

Covariance Spectral Density

SPD: Positive stiffness

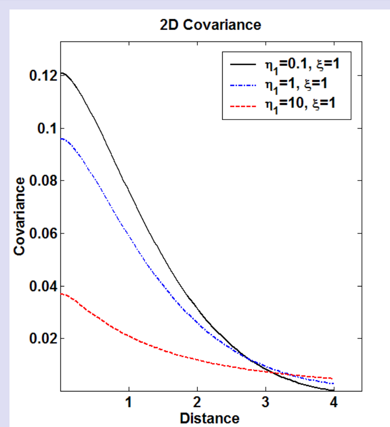


SPD: Negative stiffness

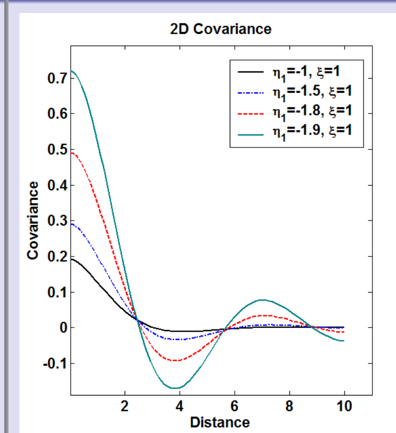


Covariance functions

Covariance ($d = 2$): Positive stiffness



Covariance ($d = 2$): Negative stiffness



Spectral Representation of Covariance Functions

Spectral Representation (Inverse Hankel transform)

- For isotropic covariance functions the following holds: *[Schoenberg, 1938]*

$$G(\mathbf{r}) = \frac{\eta_0\, \xi^d\, \|\mathbf{r}\|}{(2\pi\|\mathbf{r}\|)^{d/2}} \int_0^{k_c} d\kappa \, \frac{\kappa^{d/2} J_{d/2-1}(\kappa\|\mathbf{r}\|)}{1+\eta_1(\kappa\xi)^2+(\kappa\xi)^4}$$

- $J_{d/2-1}(\|\mathbf{r}\|)$: *Bessel function of the first kind of order $d/2-1$*
- For $k_c \to \infty$ (limit of infinite UV cutoff) the spectral integral exists for $d \leq 3$

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SSRF Covariance function $d = 1$

Unlimited band, $k_c \to \infty$ *[Hristopulos and Elogne, 2007]*

$$G(h) = \frac{\eta_0}{4}\, e^{-h\beta_2} \left[\frac{\cos(h\beta_1)}{\beta_2} + \frac{\sin(h\beta_1)}{\beta_1} \right], \quad |\eta_1| < 2$$

$$G(h) = \eta_0 \, \frac{(1+h)}{4\, e^h}, \quad \eta_1 = 2$$

$$G(h) = \frac{\eta_0}{2\,\Delta} \left(\frac{e^{-h\,\omega_1}}{\omega_1} - \frac{e^{-h\,\omega_2}}{\omega_2} \right), \quad \eta_1 > 2$$

- $h = |r|/\xi$: normalized lag,
- $\beta_{1,2} = \left(\frac{|2\mp\eta_1|}{4}\right)^{1/2}, \quad \omega_{1,2} = \left(\frac{|\eta_1\mp\Delta|}{2}\right)^{1/2}, \quad \Delta = |\eta_1^2 - 4|^{\frac{1}{2}}$

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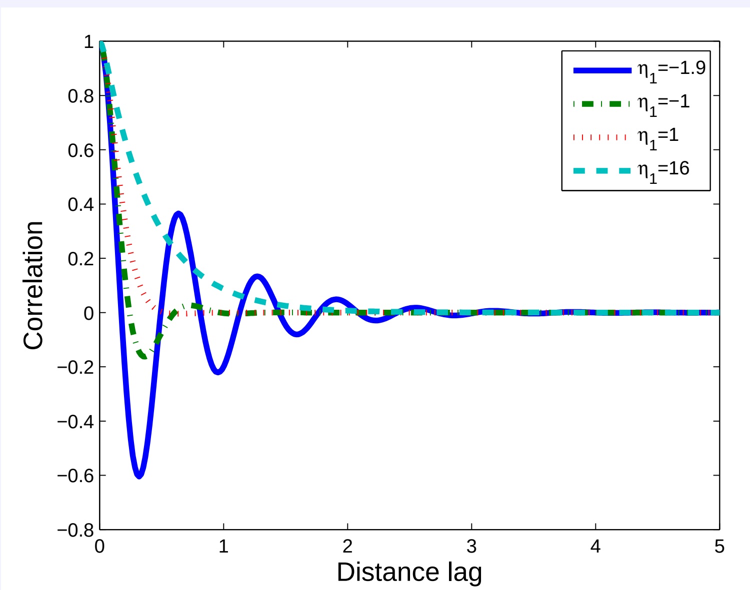

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SSRF Covariance function $d = 2$

Unlimited band, $k_c \to \infty$ *[Hristopulos, 2013]*

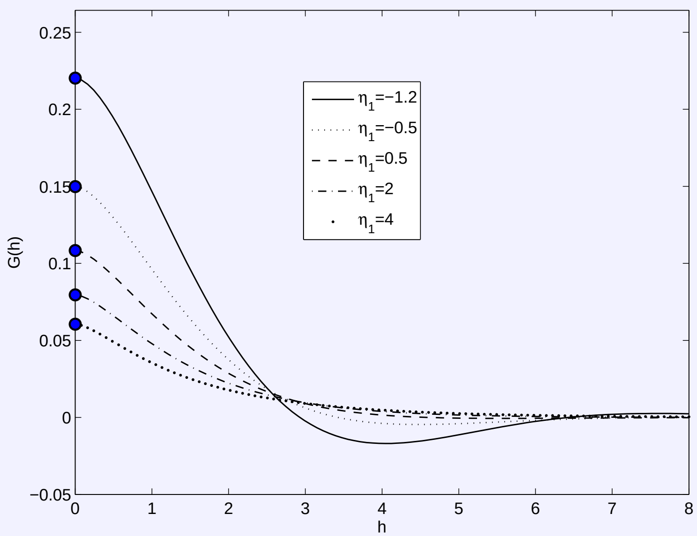
$$G(h) = \frac{\eta_0\, \Im\left[K_0(h\,z_+)\right]}{\pi\sqrt{4-\eta_1^2}}, \quad |\eta_1| < 2$$

$$G(h) = \left(\frac{\eta_0\, h}{4\pi}\right) K_{-1}(h), \quad \eta_1 = 2$$

$$G(h) = \frac{\eta_0\left[K_0(h\,z_+) - K_0(h\,z_-)\right]}{2\pi\sqrt{\eta_1^2-4}}, \quad \eta_1 > 2$$

- $h = \|\mathbf{r}\|/\xi, \quad \Im$: Imaginary part
- $z_\pm = \sqrt{-t_\pm^*}, \quad t_\pm^* = \left(-\eta_1 \pm \sqrt{\eta_1^2 - 4}\right)/2$
- $K_\nu(z)$: *modified Bessel function of the second kind and order ν*

SSRF Covariance function $d = 2$



SSRF Covariance function $d = 3$

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Unlimited band, $k_c \to \infty$ *[Hristopulos and Elogne, 2007]*

$$G(h) = \eta_0 \, \frac{e^{-h\beta_2}}{\Delta} \left[\frac{\sin(h\beta_1)}{h} \right], \quad |\eta_1| < 2$$

$$G(h) = \frac{\eta_0}{4} \, e^{-h}, \quad \eta_1 = 2$$

$$G(h) = \frac{1}{2\,\Delta} \left(\frac{e^{-h\omega_1} - e^{-h\omega_2}}{h} \right), \quad \eta_1 > 2$$

- $h = \|\mathbf{r}\|/\xi$,
- $\beta_{1,2} = \left(\frac{|2 \mp \eta_1|}{4} \right)^{1/2}, \quad \omega_{1,2} = \left(\frac{|\eta_1 \mp \Delta|}{2} \right)^{1/2}, \quad \Delta = |\eta_1^2 - 4|^{\frac{1}{2}}$

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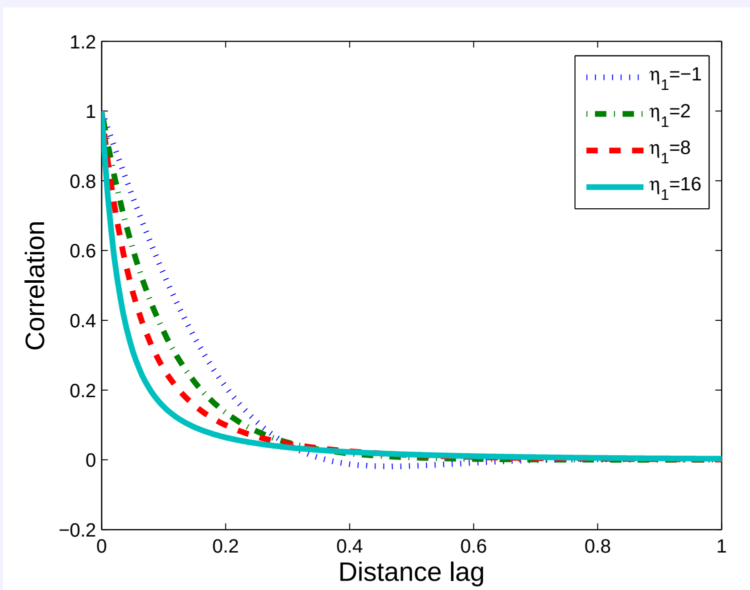

FGC-SSRF Realizations $d = 1$

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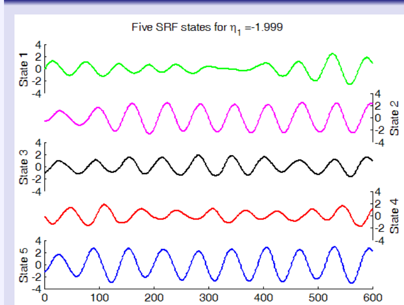
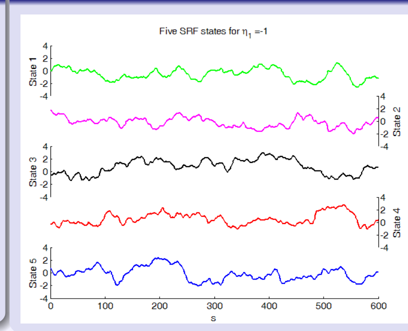
$\eta_1 = -1.999$

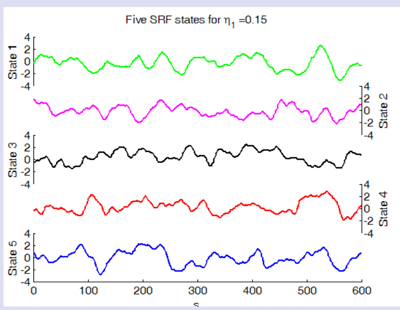


$\eta_1 = -1$

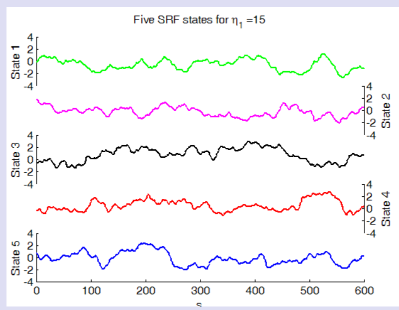


FGC-SSRF Realizations $d = 1$

$\eta_1 = 0.15$



$\eta_1 = 15$



FGC-SSRF Length Scales

FGC-SSRF covariance functions have non-linear dependence of correlation scales on model parameters *[Hristopulos and Žukovič, 2011]*

Definitions

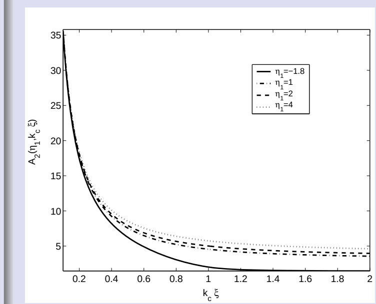
1. Integral range:

$$\ell_c \doteq \left[\frac{\int d\mathbf{r}\, G(r)}{G(0)} \right]^{1/d} = A_d\, \xi$$

2. Correlation length:

$$r_c \doteq \left[\frac{\int d\mathbf{r}\, r^2\, G(r)}{\int d\mathbf{r}\, G(r)} \right]^{1/2} = \sqrt{\left[\left| \frac{d^2 \tilde{G}(k)/dk^2}{2\tilde{G}(k)} \right|_{k=0} \right]} = \sqrt{|\eta_1|}\, \xi$$

FGC-SSRF Integral Range



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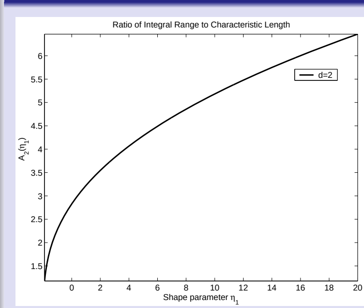
FGC-SSRF Integral Range



Karhunen-Loève Expansions of SSRFs

Karhunen-Loève Theorem

1. A second-order $\phi(\mathbf{s})$ with continuous covariance covariance $G(\mathbf{s}, \mathbf{s}')$ can be expanded on a closed and bounded domain $\mathcal{D}$ as:

$$\phi(\mathbf{s}) = m_x(\mathbf{s}) + \sum_{m=1}^{\infty} \sqrt{\lambda_m}\, c_m\, \psi_m(\mathbf{s}).$$

   The convergence is uniform on $\mathcal{D}$.

2. The λ_m and $\psi_m(\mathbf{s})$ are respectively, *eigenvalues and eigenfunctions* of the covariance operator, that satisfy the Fredholm integral equation

$$\int_{\mathcal{D}} d\mathbf{s}'\, G(\mathbf{s}, \mathbf{s}')\, \psi_m(\mathbf{s}') = \lambda_m\, \psi_m(\mathbf{s}').$$

3. The c_m are *zero-mean, uncorrelated random variables*, i.e, $\mathbb{E}[c_m] = 0$ and $\mathbb{E}[c_m\, c_n] = \delta_{n,m}$, $\forall n, m \in \mathbb{N}$.

Karhunen-Loève Expansions of SSRFs

Simulations on Square Domain

- Square lattice 100×100, Modes used: 100×100 wave-vectors
- Pinned boundaries (no fluctuations)
- SSRF eigenvalues

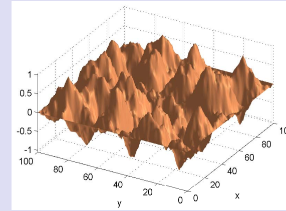
$$\lambda_m = \frac{\eta_0 \xi^2}{1 + \eta_1 \xi^2 \|\mathbf{k}_m\|^2 + \xi^4 \|\mathbf{k}_m\|^4}, \quad \mathbf{k}_m = \left(\frac{2\pi n_{m,1}}{L}, \frac{2\pi n_{m,2}}{L} \right)^T$$

- SSRF K-L eigenfunctions

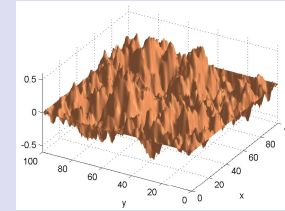
$$\psi_m(s_1, s_2) = \frac{1}{L} \sin(k_{m,1}\, s_1) \sin(k_{m,2}\, s_2),$$

Karhunen-Loève Expansions of SSRFs

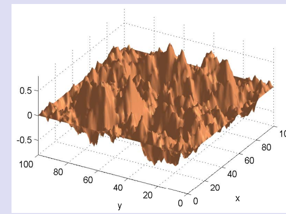
$\eta_0 = 2,\ \eta_1 = -1.5,\ \xi = 5$



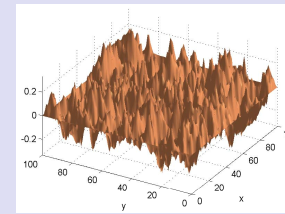
$\eta_0 = 2,\ \eta_1 = 1.5,\ \xi = 5$



$\eta_0 = 2,\ \eta_1 = 0,\ \xi = 5$



$\eta_0 = 2,\ \eta_1 = 15,\ \xi = 5$



Karhunen-Loève Expansions of SSRFs

SRRF Variance Evolution versus number of ordered eigenvalues

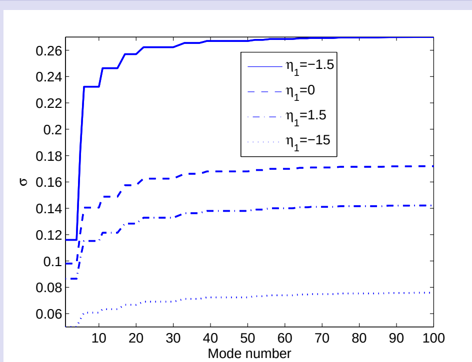


Figure: SSRF standard deviation of K-L simulation at $\mathbf{s}_0 = (25, 25)$ on 100×100 square domain with pinned boundaries. SSRF parameters: $\eta_0 = 2$, $\xi = 5$ and $\eta_1 = (-1.5, 0, 1.5, 15)^T$.

SSRF Inverse Covariance Kernel with Ultraviolet Cutoff

- Spectral integral of $\tilde{\mathbb{J}}_S(\|\mathbf{k}\|) = (\eta_0 \xi^d)^{-1} \left[1 + \eta_1 (\xi \|\mathbf{k}\|)^2 + c_2 (\xi \|\mathbf{k}\|)^4\right]$,

$$\mathbb{J}_S(\mathbf{r}; \boldsymbol{\theta}) = \frac{\|\mathbf{r}\|}{(2\pi \|\mathbf{r}\|)^{d/2}} \int_0^{k_c} d\|\mathbf{k}\|\, \|\mathbf{k}\|^{d/2} J_{d/2-1}(\|\mathbf{k}\| \|\mathbf{r}\|)\, \tilde{\mathbb{J}}_S(\|\mathbf{k}\|; \boldsymbol{\theta}),$$

- Lommel functions

$$z^2 \frac{d^2 w(z)}{dz^2} + z \frac{dw(z)}{dz} + (z^2 - \nu^2)\, w(z) = z^{\mu+1}.$$

- If $\mu - \nu = 2l + 1$ the following Lommel series terminates after $l + 1$ terms

$$S_{\mu,\nu}(z) = z^{\mu-1} \left[1 - \frac{(\mu-1)^2 - \nu^2}{z^2} + \frac{[(\mu-1)^2 - \nu^2][(\mu-3)^2 - \nu^2]}{z^4} - \ldots \right]$$

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In $d \geq 2$, the SSRF inverse covariance kernel $\mathbb{J}_S(z;\boldsymbol{\theta})$ is given by means of the following tripartite sum, where $u_c = k_c\xi$, $z = k_c\,\|\mathbf{r}\|$, and $\nu = d/2 - 1$

$$\mathbb{J}_S(z;\boldsymbol{\theta}) = \sum_{l=0,1,2} \frac{g_l(\boldsymbol{\theta})}{z^{2\nu+2l+1}} \left[(2\nu+2l)J_\nu(z)\,S_{\nu+2l,\nu-1}(z) - J_{\nu-1}(z)\,S_{\nu+2l+1,\nu}(z)\right],$$

$$g_0(\boldsymbol{\theta}) = \frac{k_c^d}{(2\pi)^{d/2}\eta_0\,\xi^d},\; g_1(\boldsymbol{\theta}) = \eta_1\,u_c^2\,g_0(\boldsymbol{\theta}),\; g_2(\boldsymbol{\theta}) = u_c^4\,g_0(\boldsymbol{\theta}),$$

The above equations define a positive semidefinite kernel function for $\eta_1 > -2$. In particular, the value at the origin is

$$\mathbb{J}_S(0;\boldsymbol{\theta}) = \frac{g_0(\boldsymbol{\theta})}{2^{\nu+1}\,\Gamma(\nu+2)}\left[1 + \eta_1\,u_c^2\binom{\nu+1}{\nu+2} + u_c^4\binom{\nu+1}{\nu+3}\right],$$

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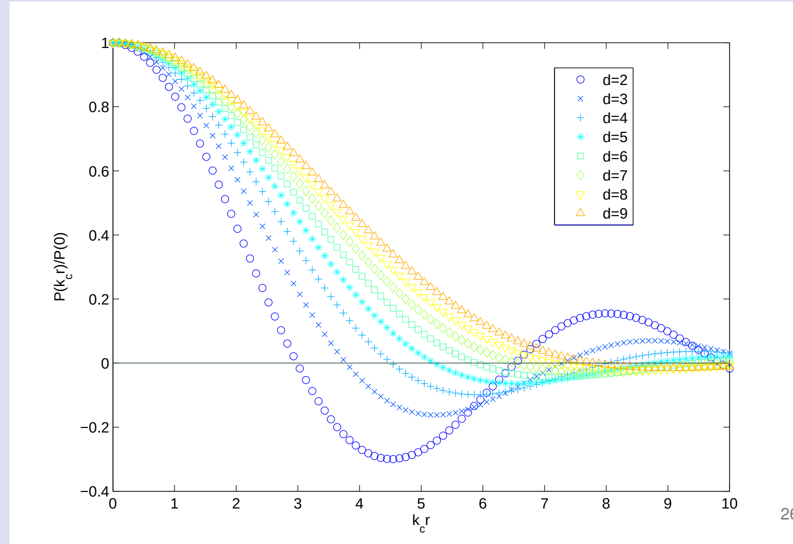
Normalized inverse SSRF covariance function vs d

SSRF parameters are $\eta_0 = 1$, $\xi = 1$, $\eta_1 = 2$, and $k_c = 2$.



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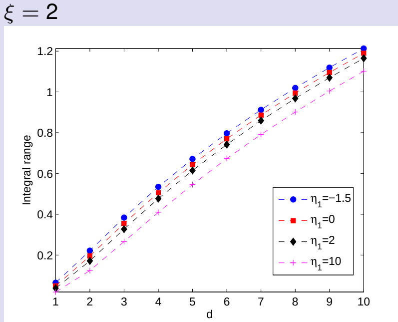
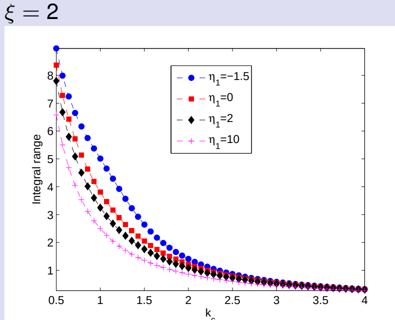
Integral range ℓ_c

Dependence on d for different η_1 and $\xi = 2$



Integral range ℓ_c

Dependence on k_c for different η_1 and $\xi = 2$



Fast optimal interpolation

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Theorem

Let $\mathbf{X}_s = (X_1,\ldots,X_N)^T$ a vector of measurements at $\mathbf{s}_n$, $n = 1,\ldots,N$, and $X_p = X_{N+1}$ the SSRF value at unmeasured location $\mathbf{s}_{N+1}$. Assume that the data are samples of the SSRF with the energy functional

$$H_{\rm fgc}[\mathbf{X}_{s;p};\theta] = \frac{1}{2}\,\mathbf{X}_{s;p}^T\,\mathbf{J}(\theta)\,\mathbf{X}_{s;p},$$

where $\mathbf{X}_{s;p} = (X_1,\ldots,X_N,X_p)^T$, and $\mathbf{J}(\boldsymbol{\theta})$ is the inverse covariance (precision) matrix. The mode estimate $\hat{X}_p$ which maximizes the joint pdf is given by

$$\hat{X}_p = -\frac{\mathbf{J}_{p;s}^T(\boldsymbol{\theta})\,\mathbf{X}_s}{J_{p;p}(\boldsymbol{\theta})} = -\sum_{i=1}^{N}\frac{J_{p,i}(\boldsymbol{\theta})\,X_i}{J_{p;p}(\boldsymbol{\theta})}. \qquad (1)$$

Grid Approximation of Continuum SSRF

$$\mathbb{J}_S(\mathbf{r_n};\boldsymbol{\theta}) = c_0 \left[1 - \eta_1 \left(\sum_{i=1}^{d} \frac{\xi^2}{a_i^2} \mathrm{D}^2_{\mathbf{n},i} \right) + \left(\sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\xi^4}{a_i^2\, a_j^2} \mathrm{D}^2_{\mathbf{n},i} \mathrm{D}^2_{\mathbf{n},j} \right) \right] \delta(\mathbf{r_n}).$$

On a grid with lattice step $a \to 0$: $\delta(\mathbf{s}_i - \mathbf{s}_j) \to \delta_{i,j}/v_c$, $v_c = a^d$.

$$\delta_{\mathbf{n},i}[f(\mathbf{r_n})] = f\left(\mathbf{r}_n + \frac{a_i}{2}\,\hat{\mathbf{e}}_i\right) - f\left(\mathbf{r}_n - \frac{a_i}{2}\,\hat{\mathbf{e}}_i\right),\ i = 1, \ldots, d.$$

$$\delta^2_{\mathbf{n},i}[f(\mathbf{r_n})] = f(\mathbf{r}_n + a_i\,\hat{\mathbf{e}}_i) + f(\mathbf{r}_n - a_i\,\hat{\mathbf{e}}_i) - 2f(\mathbf{r}_n).$$

$\mathrm{D}_{\mathbf{n},i}$ is related to the centered difference operator by *[Hildebrand, 1974]*

$$a_i\,\mathrm{D}_{\mathbf{n},i} = 2\sinh^{-1}\left(\frac{\delta_{\mathbf{n},i}}{2}\right).$$

Taylor series expansions of $\mathrm{D}^2_{\mathbf{n},i}$ and $\mathrm{D}^4_{\mathbf{n},i}$ in terms of $\delta_{\mathbf{n},i}$

$$a_i^2\,\mathrm{D}^2_{\mathbf{n},i} = \delta^2_{\mathbf{n},i} - \frac{\delta^4_{\mathbf{n},i}}{12} + \frac{\delta^6_{\mathbf{n},i}}{90} - \frac{\delta^8_{\mathbf{n},i}}{560} + \frac{\delta^{10}_{\mathbf{n},i}}{3150} - \frac{\delta^{12}_{\mathbf{n},i}}{16632} + O\left(\delta^{14}_{\mathbf{n},i}\right)$$

$$a_i^4\,\mathrm{D}^4_{\mathbf{n},i} = \delta^4_{\mathbf{n},i} - \frac{\delta^6_{\mathbf{n},i}}{6} + \frac{7\delta^8_{\mathbf{n},i}}{240} - \frac{41\delta^{10}_{\mathbf{n},i}}{7560} + \frac{479\delta^{12}_{\mathbf{n},i}}{453600} + O\left(\delta^{14}_{\mathbf{n},i}\right).$$

Inverse SSRF Covariance Kernel on Hypercubic Grid

$$\mathbb{J}(\mathbf{r_n};\boldsymbol{\theta}) = c_0 \left[\delta_{\mathbf{n},\mathbf{0}} - \eta_1\,\xi^2\,S(\mathbf{r_n}) + \xi^4\,C(\mathbf{r_n}) \right],$$

where the square gradient, $S(\mathbf{r_n})$, and square curvature, $C(\mathbf{r_n})$, are given by

$$S(\mathbf{r_n}) = \sum_{i=1}^{d} \mathrm{D}^2_{\mathbf{n},i}, \quad C(\mathbf{r_n}) = \sum_{i=1}^{d} \mathrm{D}^4_{\mathbf{n},i} + \sum_{i=1}^{d}\sum_{j=1}^{d} \mathrm{D}^2_{\mathbf{n},i}\,\mathrm{D}^2_{\mathbf{n},j}.$$

Truncating $S(\mathbf{r_n})$ and $C(\mathbf{r_n})$ at order $2p = 12$, it follows that

$$S^{(12)}(\mathbf{r_n}) = \sum_{i=1}^{d} \delta^2_{\mathbf{n},i} - \frac{\delta^4_{\mathbf{n},i}}{12} + \frac{\delta^6_{\mathbf{n},i}}{90} - \frac{\delta^8_{\mathbf{n},i}}{560} + \frac{\delta^{10}_{\mathbf{n},i}}{3150} - \frac{\delta^{12}_{\mathbf{n},i}}{16632},$$

$$C^{(12)}(\mathbf{r_n}) = \sum_{i=1}^{d} \left(\delta^4_{\mathbf{n},i} - \frac{\delta^6_{\mathbf{n},i}}{6} + \frac{7\,\delta^8_{\mathbf{n},i}}{240} - \frac{41\,\delta^{10}_{\mathbf{n},i}}{7560} + \frac{479\,\delta^{12}_{\mathbf{n},i}}{453600} \right) + \sum_{i=1}^{d}\sum_{j=1}^{d} \left(\delta^2_{\mathbf{n},i}\delta^2_{\mathbf{n},j} - \frac{\delta^2_{\mathbf{n},i}\,\delta^4_{\mathbf{n},j}}{6} + \frac{\delta^2_{\mathbf{n},i}\delta^6_{\mathbf{n},j}}{45} - \frac{\delta^2_{\mathbf{n},i}\delta^8_{\mathbf{n},j}}{280} + \frac{\delta^2_{\mathbf{n},i}\delta^{10}_{\mathbf{n},j}}{1575} + \frac{\delta^4_{\mathbf{n},i}\,\delta^4_{\mathbf{n},j}}{144} - \frac{\delta^4_{\mathbf{n},i}\,\delta^6_{\mathbf{n},j}}{540} + \frac{\delta^4_{\mathbf{n},i}\,\delta^8_{\mathbf{n},j}}{3360} + \frac{\delta^6_{\mathbf{n},i}\,\delta^6_{\mathbf{n},j}}{8100} \right).$$

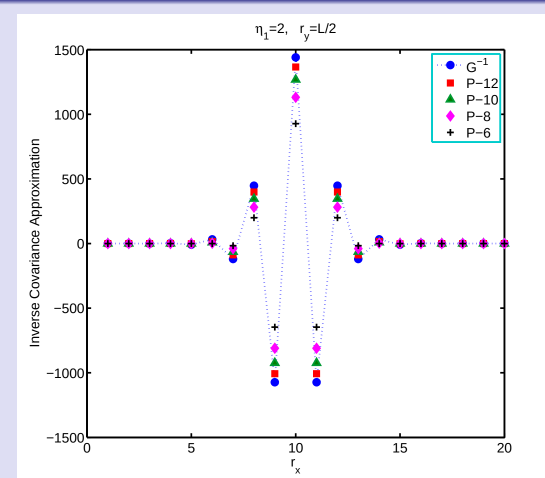
Inverse SSRF Covariance Kernel on Hypercubic Grid

Table: Central finite differences of orders $2k$, $k = 1, \ldots, 6$ on hypercubic grid with uniform step $a = 1$ in each orthogonal direction $i = 1, \ldots, d$. FD stands for *finite difference*. The $f_{\mathbf{n}}$ denotes any lattice function.

FD	Expressions
$\delta^2_{\mathbf{n},i} f_{\mathbf{n}} =$	$f_{\mathbf{n}+\hat{\mathbf{e}}_i} + f_{\mathbf{n}-\hat{\mathbf{e}}_i} - 2f_{\mathbf{n}}$
$\delta^4_{\mathbf{n},i} f_{\mathbf{n}} =$	$f_{\mathbf{n}+2\hat{\mathbf{e}}_i} + f_{\mathbf{n}-2\hat{\mathbf{e}}_i} - 4f_{\mathbf{n}+\hat{\mathbf{e}}_i} - 4f_{\mathbf{n}-\hat{\mathbf{e}}_i} + 6f_{\mathbf{n}}$
$\delta^6_{\mathbf{n},i} f_{\mathbf{n}} =$	$f_{\mathbf{n}+3\hat{\mathbf{e}}_i} + f_{\mathbf{n}-3\hat{\mathbf{e}}_i} - 6f_{\mathbf{n}+2\hat{\mathbf{e}}_i} - 6f_{\mathbf{n}-2\hat{\mathbf{e}}_i} + 15f_{\mathbf{n}+\hat{\mathbf{e}}_i} + 15f_{\mathbf{n}-\hat{\mathbf{e}}_i} - 20f_{\mathbf{n}}$
$\delta^8_{\mathbf{n},i} f_{\mathbf{n}} =$	$f_{\mathbf{n}+4\hat{\mathbf{e}}_i} + f_{\mathbf{n}-4\hat{\mathbf{e}}_i} - 8f_{\mathbf{n}+3\hat{\mathbf{e}}_i} - 8f_{\mathbf{n}-3\hat{\mathbf{e}}_i} + 28f_{\mathbf{n}+2\hat{\mathbf{e}}_i} + 28f_{\mathbf{n}-2\hat{\mathbf{e}}_i} - 56f_{\mathbf{n}+\hat{\mathbf{e}}_i}$ $-56f_{\mathbf{n}-\hat{\mathbf{e}}_i} + 70f_{\mathbf{n}}$
$\delta^{10}_{\mathbf{n},i} f_{\mathbf{n}} =$	$f_{n_i+5,n_j} + f_{n_i-5,n_j} - 10f_{\mathbf{n}+4\hat{\mathbf{e}}_i} - 10f_{\mathbf{n}-4\hat{\mathbf{e}}_i} + 45f_{\mathbf{n}+3\hat{\mathbf{e}}_i} + 45f_{\mathbf{n}-3\hat{\mathbf{e}}_i} - 120f_{\mathbf{n}+2\hat{\mathbf{e}}_i}$ $-120f_{\mathbf{n}-2\hat{\mathbf{e}}_i} + 210f_{\mathbf{n}+\hat{\mathbf{e}}_i} + 210f_{\mathbf{n}-\hat{\mathbf{e}}_i} - 252f_{\mathbf{n}}$
$\delta^{12}_{\mathbf{n},i} f_{\mathbf{n}} =$	$f_{n_i+6,n_j} + f_{n_i-6,n_j} - 12f_{n_i+5,n_j} - 12f_{n_i-5,n_j} + 66f_{\mathbf{n}+4\hat{\mathbf{e}}_i} + 66f_{\mathbf{n}-4\hat{\mathbf{e}}_i} - 220f_{\mathbf{n}+3\hat{\mathbf{e}}_i}$ $-220f_{\mathbf{n}-3\hat{\mathbf{e}}_i} + 495f_{\mathbf{n}+2\hat{\mathbf{e}}_i} + 495f_{\mathbf{n}-2\hat{\mathbf{e}}_i} - 792f_{\mathbf{n}+\hat{\mathbf{e}}_i} - 792f_{\mathbf{n}-\hat{\mathbf{e}}_i} + 924f_{\mathbf{n}}$

Inverse SSRF Covariance Kernel on Hypercubic Grid

Numerical Example



Details

Inverse SSRF covariance kernel $G^{-1}(L/2, k)$, $k = 1, \ldots, L$ evaluated by inverting the $d = 1$ continuum covariance (circles) versus the centered FD inverse kernel expressions $\mathbb{J}^{(2p)}(L/2, k)$, $p = 3, 4, 5, 6$ of orders six (crosses), eight (diamonds), ten (triangles), and twelve (squares). An 1D chain of length $L = 20$ ($a = 1$) is used. SSRF parameters: $\eta_0 = 10$, $\eta_1 = 2$, and $\xi = 10$. The covariance is given by $G(h) = \eta_0(1 + h)\mathrm{e}^{-h}/4$, where $h = |r|/\xi$, *[Hristopulos and Elogne, 2007]*.

Conclusions and Future Directions

- We presented three-parameter, positive-definite, isotropic covariance models based on local interaction "energy" (Spartan) functionals
- SSRF models lead to fast (linear complexity) interpolation on regular grids and on unstructured grids as well *[Hristopulos and Elogne, 2009]*
- A new family of four-parameter, positive-definite kernels valid in $d \geq 2$ based on Lommel functions is proposed
- Continuing research involves extensions to: spatial non-homogeneity, space-time correlations, and non-Gaussian dependence

SSRF Covariance Models

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SSRF Covariance Functions
KL Expansions
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